Calculating and drawing Belyi pairs

George SHABAT

Russian State University for the Humanities,
Moscow State Iniversity, ITEP

...il y a une identité profonde entre la combinatoire des cartes finies d'une part, et la géometrie des courbes algébriques définies sur des corps de nombres, de l'autre. Ce résultat profond, joint à l'interprétation algébrico-géometrique des cartes finies, ouvre la porte sur un monde nouveau, inexploré – et à porté de main de tous qui passent sans le voir.

Category equivalencies

$$\mathcal{DESS} \longleftrightarrow \mathcal{BELP}_2(\mathbb{C}) \simeq \mathcal{BELP}_2(\overline{\mathbb{Q}}) \longleftrightarrow \mathcal{C}_2^+\mathcal{SET}s[\ldots]$$
$$\mathcal{DESS}_3 \longleftrightarrow \mathcal{BELP}(\mathbb{C}) \simeq \mathcal{BELP}(\overline{\mathbb{Q}}) \longleftrightarrow (\tfrac{\mathbb{Z}}{2\mathbb{Z}} * \tfrac{\mathbb{Z}}{2\mathbb{Z}} * \tfrac{\mathbb{Z}}{2\mathbb{Z}})\mathcal{SET}s[\ldots]$$

Object-by-object correspondence?

Reasons to study

- Fun: observable;
- Defining and comparing *complexities* (theory by Kolmogorov-Zvonkin-Levin-...): first steps, see recent papers by Javanpeykar, Litcanu, Zapponi;
- Transferring structures

from $\mathcal{DESS}$ to $\mathcal{BELP}$ (edge contraction $\longleftrightarrow$??,
Zograf enumeration $\longleftrightarrow$??),
from $\mathcal{BELP}$ to $\mathcal{DESS}$ (discriminants of fields of definition??,
primes of bad reduction?,
Aut$\overline{\mathbb{Q}}$- ORBITS??)–
dreams...

A brief historical overview: pre-Grothendieck era

• platonic solids: see Klein's "Icosahedron" for $\{\beta^{-1\circ}\}$;

• generalization: *Schwartz list*;

• classical symmetric curves: Klein quartic (see Elkies), Bring curve (listen Zvonkin), Fricke curve (look at the conference poster...),...

$$\beta : C \longrightarrow \frac{C}{\mathrm{Aut}C} \simeq \mathbf{P}_1(\mathbb{C})$$

• Cayley graphs...

...

Grothendieck era: few examples

- 5-trees/$\mathbb{Q}$, 6-trees/ $\mathbb{Q}(\sqrt[3]{2})$ (Sh-Voevodsky, Drinfel'd);
- Leila's flower (L. Schneps, potential 24-orbit split into 12+12)
- Mathieu trees: ERG's=$\mathsf{M}_{11,23}$ (Yu.V. Matijasevich)
- Toric pan (V.Dremov)
- A cubic 4-edged toric orbit (Sh)

Small degree

Catalogs:

- All (clean) dessins with ≤ 3 edges (Sh, 1991)
- All trees with ≤ 8 edges (Bétréma, Péré, Zvonkin, 1992)
- All (clean) dessins with ≤ 4 edges (Adrianov, ..., 2007)
- All trees with 9 edges (Kochetkov, 2008)

Algebraic tricks

• **Left-composing β wih smth nice.** E.g., $\mathrm{Juk} \circ \beta := \frac{1}{2}(\beta + \frac{1}{\beta})$ was used by Adrianov, Bychkov, Dremov, Epifanov, Oganesjan in several calculations.

• **Mulase-Penkava operator.**

$$\mathrm{MP}(\beta) := \frac{(\mathrm{d}\beta)^2}{\beta^2(1-\beta)}$$

It is Strebel! Related to metrised ribbon graphs.

Some other methods

See Sijsling, Voight (2014) for an overview – 176 refs

- Reductions $\to$ lifting to p-adic Belyi pairs $\to$ LLL;
- Approximate calculations (circle packing, discrete periods,...);
- Algebraic solutions of differential equations (Painleve-6, Heun,...);
- ...

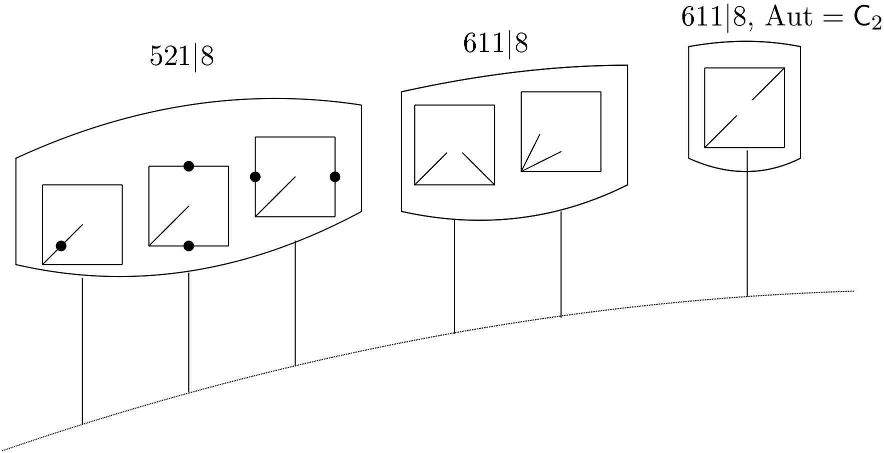
Relaxing $\#\{\text{BranchPoints}\} \leq 3$ to ≤ 4

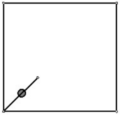
Replace

- 3 by 4;
- *Belyi* by *Fried*;
- *curves* by *families*.

New way of calculating Belyi functions: colliding critical values in the Fried families. Putting everything into moduli spaces.

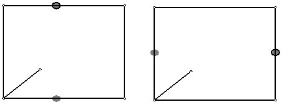
Example: several Belyi pairs on a Fried curve





One cubic Galois orbit

The ***j***-invariant is the real root of I_{521}



The ***j***-invariants are non-real roots of I_{521}

$$I_{521} = 56495049800000000000000 j^3 - \dots$$

$$56495049800000000000000 = 2^{15} \cdot 5^{14} \cdot 7^{10}$$

Open problems

Families?
Asymptotics?
Homology of critical strata!??
...